Half-angle identities

In this lesson we're looking at the half-angle identities for sine and cosine. We derive them from the basic Pythagorean identity and the double angle identities. The half-angle identities are

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

Sometimes the half-angle formulas are expressed as formulas for $\cos\theta$ and $\sin\theta$ in terms of 2θ (instead of as formulas for $\cos(\theta/2)$ and $\sin(\theta/2)$ in terms of θ). To do this, we simply take the half-angle identities we've already found and substitute 2θ for θ:

$$\cos\theta = \pm\sqrt{\frac{1+\cos 2\theta}{2}}$$

$$\sin\theta = \pm\sqrt{\frac{1-\cos 2\theta}{2}}$$

Example

If $\pi < \theta < 3\pi/2$ and $\cos\theta = -\sqrt{6}/7$, find $\cos(\theta/2)$ and $\sin(\theta/2)$.

By the half-angle identity for cosine,

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$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

Substituting $-\sqrt{6}/7$ for $\cos\theta$ gives

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\left(-\frac{\sqrt{6}}{7}\right)}{2}} = \pm\sqrt{\frac{\frac{7(1)+1\left(-\sqrt{6}\right)}{7}}{2}} = \pm\sqrt{\frac{7-\sqrt{6}}{14}}$$

By the half-angle identity for sine,

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

Substituting $-\sqrt{6}/7$ for $\cos\theta$ gives

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\left(-\frac{\sqrt{6}}{7}\right)}{2}} = \pm\sqrt{\frac{\frac{7(1)+1\sqrt{6}}{7}}{2}} = \pm\sqrt{\frac{7+\sqrt{6}}{14}}$$

We've been told that θ is in the third quadrant, which means that $\theta/2$ has to be in the second quadrant. Therefore, $\cos(\theta/2)$ is negative and $\sin(\theta/2)$ is positive, so

$$\cos\frac{\theta}{2} = -\sqrt{\frac{7-\sqrt{6}}{14}}$$

$$\sin\frac{\theta}{2} = \sqrt{\frac{7+\sqrt{6}}{14}}$$

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Next we'll consider an angle that isn't in the interval $[0,2\pi)$.

Example

If $-11\pi/2 < \theta < -5\pi$ and $\sin\theta = 1/3$, find $\cos(\theta/2)$ and $\sin(\theta/2)$.

Since the half-angle identities for both cosine and sine have the quantity "$\cos\theta$" in them, we first need to determine the value of $\cos\theta$, so we'll use the basic Pythagorean identity.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

Substituting $1/3$ for $\sin\theta$ gives

$$\cos^2\theta = 1 - \left(\frac{1}{3}\right)^2$$

$$\cos^2\theta = 1 - \frac{1}{9}$$

$$\cos^2\theta = \frac{8}{9}$$

We were told that $-11\pi/2 < \theta < -5\pi$, which means that θ is in the second quadrant. The cosine of every angle in the second quadrant is negative, so

$$\cos\theta = -\sqrt{\frac{8}{9}} = -\frac{\sqrt{8}}{\sqrt{9}} = -\frac{2\sqrt{2}}{3}$$

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By the half-angle identity for cosine,

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

Substituting $-2\sqrt{2}/3$ for $\cos\theta$ gives

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\left(-\frac{2\sqrt{2}}{3}\right)}{2}} = \pm\sqrt{\frac{\frac{3(1)+1\left(-2\sqrt{2}\right)}{3}}{2}} = \pm\sqrt{\frac{3-2\sqrt{2}}{6}}$$

By the half-angle identity for sine,

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

Substituting $-2\sqrt{2}/3$ for $\cos\theta$ gives

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\left(-2\frac{\sqrt{2}}{3}\right)}{2}} = \pm\sqrt{\frac{\frac{3(1)+1\left(2\sqrt{2}\right)}{3}}{2}} = \pm\sqrt{\frac{3+2\sqrt{2}}{6}}$$

Earlier, we found that θ was in the second quadrant. Because we're dealing with a negative angle, that means $\theta/2$ has to be in the third quadrant. Both the cosine function and the sine function are negative for all angles in the third quadrant, so

$$\cos\frac{\theta}{2} = -\sqrt{\frac{3-2\sqrt{2}}{6}}$$

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$$\sin\frac{\theta}{2} = -\sqrt{\frac{3+2\sqrt{2}}{6}}$$

Even if we're given the value of $\tan\theta$ for some angle θ and the interval in which it lies, we can find $\cos(\theta/2)$ and $\sin(\theta/2)$.

Example

Find $\sin(\theta/2)$ and $\cos(\theta/2)$ for the angle θ such that $\tan\theta = 21$ and θ lies in the interval $(10\pi, 10\pi + (\pi/2))$.

If we start with the Pythagorean identity

$$\sec^2\theta = 1 + \tan^2\theta$$

and substitute 21 for $\tan\theta$, we get

$$\sec^2\theta = 1 + (21^2) = 1 + 441 = 442$$

By the reciprocal identity

$$\cos\theta = \frac{1}{\sec\theta}$$

we get

$$\cos^2\theta = \frac{1}{\sec^2\theta} = \frac{1}{442}$$

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We've been told that

$$10\pi < \theta < 10\pi + \frac{\pi}{2}$$

Since 10π is an integer multiple of 2π, we see that θ is in the first quadrant, so $\cos\theta$ is positive.

$$\cos\theta = \frac{1}{\sqrt{442}}$$

By the half-angle identity for cosine,

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

Substituting $1/\sqrt{442}$ for $\cos\theta$ gives

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\frac{1}{\sqrt{442}}}{2}} = \pm\sqrt{\frac{\frac{\sqrt{442}}{\sqrt{442}}+\frac{1}{\sqrt{442}}}{2}} = \pm\sqrt{\frac{\frac{1+\sqrt{442}}{\sqrt{442}}}{2}} = \pm\sqrt{\frac{1+\sqrt{442}}{2\sqrt{442}}}$$

By the half-angle identity for sine,

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

Substituting $1/\sqrt{442}$ for $\cos\theta$ gives

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\frac{1}{\sqrt{442}}}{2}} = \pm\sqrt{\frac{\frac{\sqrt{442}}{\sqrt{442}}-\frac{1}{\sqrt{442}}}{2}} = \pm\sqrt{\frac{\frac{\sqrt{442}-1}{\sqrt{442}}}{2}} = \pm\sqrt{\frac{\sqrt{442}-1}{2\sqrt{442}}}$$

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We've been told that

$$10\pi < \theta < 10\pi + \frac{\pi}{2}$$

If we divide through by 2 we get

$$5\pi < \frac{\theta}{2} < 5\pi + \frac{\pi}{4}$$

An angle of 5π lies on the negative x-axis. Thus any angle between 5π and $5\pi + \pi/4$ is in the third quadrant, so $\theta/2$ is in the third quadrant. This tell us that both $\cos\theta$ and $\sin\theta$ are negative, so

$$\cos\frac{\theta}{2} = -\pm\sqrt{\frac{1+\sqrt{442}}{2\sqrt{442}}}$$

$$\sin\frac{\theta}{2} = -\sqrt{\frac{\sqrt{442}-1}{2\sqrt{442}}}$$

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